"On some Definite Integrals and a New Method of reducing a Function of Spherical Co-ordinates to a Series of Spherical Harmonics." By ARTHUR SCHUSTER, F.R.S. Received May 30,—Read June 5, 1902.

(Abstract.)

The expansion of a function $f(\theta)$ of an angle θ varying between 0 and π in terms of a series proceeding by the sines of the multiples of θ depends on the fundamental theorem,

$$\int_0^\pi \sin p\theta \sin q\theta \, d\theta = 0,$$

where p and q are integer numbers not equal to each other. Similarly if P_n denotes the zonal harmonic of degree n, $\mu = \cos\theta$, and

$$Q_n^\sigma = \sin^\sigma\theta \frac{d^\sigma P_n}{d\mu^\sigma},$$

the expansion of a function of θ in terms of a series of the functions Q_n^σ depends on the corresponding theorem,

$$\int_{-1}^{+1} Q_n^\sigma Q_i^\sigma d\mu = 0,$$

where i and n are two integer numbers not equal to each other. In many practical applications a continuous function is given by means of its numerical values at certain points, *e.g.*, for equidistant values of θ.

Such cases present no difficulty when Fourier's analysis is to be employed, because there is in that case a summation theorem exactly corresponding to the above integration theorem. If θ be replaced by $\rho\pi/n$, where ρ takes successively the values 1, 2, 3 . . . , the equation

$$\sum_{\rho=0}^{\rho=n-1} \sin(\rho p\pi/n) \sin(\rho q\pi/n) = 0$$

will hold true. This allows us to determine the coefficients in the case of problems in which discontinuous values of the function at equidistant points are known (*e.g.*, hourly readings of temperature or barometric pressure). If we assume that all Fourier coefficients beyond the nth vanish, n equations are obtained, each of which only contains one of the unknown quantities.

If it is desired to expand a function in terms of cosines, a slight

modification must be introduced, the summation theorem in that case being

$$\tfrac{1}{2}+\tfrac{1}{2}\cos p\pi\cos q\pi+\sum_{\rho=1}^{\rho=n-1}\cos(\rho p\pi/n)\cos(\rho q\pi/n) = 0,$$

the first and second terms representing half the value of the product for $\rho = 0$ and $\rho = n$ respectively.

There is no corresponding summation theorem in the case of the functions Q_n^σ, and the application of the method of least squares leads to a series of normal equations, each of which contains *all* the other coefficients. This has been one of the great practical difficulties in obtaining an expression for the series of spherical harmonics for the earth's magnetic potential.

F. E. Neumann has tried to overcome the difficulty by calculating coefficients $a_1, a_2 \ldots a_q$ in such a way that

$$\sum_{\rho=1}^{\rho=q} a_\rho\, Q_n^\sigma(\mu_\rho)\, Q_i^\sigma(\mu_\rho) = 0.$$

Here $\mu_1, \mu_2 \ldots \mu_q$ are the quantities for which the values of the function to be represented are known. Neumann's process is equivalent to attaching weights proportional to a_ρ to the different observations, a proceeding against which theoretical objections might be urged.

2. The expansion in terms of a series of cosines and sines being so much easier than the direct expansion in terms of a series of the functions Q_n^σ, I have endeavoured to obtain the latter series by means of the former.

It is well known that a function of an angle θ, which is confined to the values lying between 0 and π, may be put either into the form

$$a_0+a_1\cos\theta+a_2\cos 2\theta+\ldots a_p\cos p\theta+\ldots,$$

or into the form

$$b_1\sin\theta+b_2\sin 2\theta+\ldots b_p\sin p\theta+\ldots$$

The reduction to the series of spherical harmonics is accomplished by calculating and tabulating the coefficients in the series

$$\cos p\theta = A_\sigma^\sigma Q_\sigma^\sigma + A_{\sigma+1}^\sigma Q_{\sigma+1}^\sigma + \ldots A_n^\sigma Q_n^\sigma + \ldots,$$

$$\sin p\theta = B_\sigma^\sigma Q_\sigma^\sigma + B_{\sigma+1}^\sigma Q_{\sigma+1}^\sigma + \ldots B_n^\sigma Q_n^\sigma + \ldots$$

The choice between the cosine and sine series is open to us, but it appears that great simplicity is gained by taking the former series when σ is odd and the latter when σ is even. For in that case the coefficients A_n^σ and B_n^σ will all vanish, as long as n is smaller than p.

When it is therefore desired to retain only terms as far as the nth degree, the Fourier coefficients need only be calculated as far as $p = n+1$. The position of the earth's magnetic axis, *e.g.*, only depending on the terms of the first degree, is completely determined by the coefficients b_2 for $\sigma = 0$ and a_0, a_2 for $\sigma = 1$.

3. The symbolical representation of the results of this paper is much facilitated by the introduction of a separate symbol for the product of alternate factors, $n \,.\, n-2 \,.\, n-4 \,.\,.\,.\, 1$, if n be odd, or $n \,.\, n-2 \,.\,.\,.\, 2$ if n be odd. I propose to write $n\,!\,!$ for such products, and if a name be required for the product to call it the "alternate factorial" or the "double factorial." Full advantage of the new symbol is only gained by extending its meaning to negative values of n. Its complete definition may then be included in the equations

$$n\,!! = n(n-2)\,!!\,, \qquad 1\,!! = 1\,, \qquad 2\,!! = 2\,.$$

From this we may derive when n is negative and odd

$$n\,!! = (-1)^{\frac{n+1}{2}} \frac{1}{(-n-2)\,!!}\,,$$

while for n negative and even, the factorial becomes infinitely large.

4. The calculation of the factors A_n^σ and B_n^σ depends on the values of the definite integrals

$$\int_{-1}^{-1} Q_n^\sigma \cos p\theta d\mu, \qquad \int_{-1}^{+1} Q_n^\sigma \sin p\theta d\mu\,,$$

and these may be made to depend on the values of the integrals

$$\int_{-1}^{+1} Q_n^\sigma \sin{}^\lambda\theta d\mu \qquad \text{and} \qquad \int_{-1}^{+1} \mu Q_n^\sigma \sin{}^\lambda\theta d\mu\,.$$

It is proved that

$$\int_{-1}^{+1} Q_n^\sigma \sin{}^\lambda\theta d\mu = c\,\frac{(n+\sigma-1)\,!!\,(\sigma+\lambda)\,!!\,(n-\lambda-2)\,!!}{(n-\sigma)\,!!\,(\sigma-\lambda-2)\,!!\,(n+\lambda+1)\,!!}\,, \text{ if } n-\sigma \text{ be even,}$$

$$= 0 \qquad \text{if } n-\sigma \text{ be odd,}$$

$$\int_{-1}^{+1} \mu Q_n^\sigma \sin{}^\lambda\theta d\mu = c\,\frac{(n+\sigma)\,!!\,(\sigma+\lambda)\,!!\,(n-\lambda-3)\,!!}{(n-\sigma-1)\,!!\,(\sigma-\lambda-2)\,!!\,(n+\lambda+2)\,!!}\,, \text{ if } n-\sigma \text{ be odd,}$$

$$= 0 \qquad \text{if } n-\sigma \text{ be even.}$$

The factor c is equal to 2 or to π according as $\sigma+\lambda$ is even or odd.

5. The integrals

$$\int_{-1}^{+1} Q_n^\sigma Q_i^\rho \, d\mu$$

are obtained in the form of a series having a finite number of terms.

6. To find

$$\int_{-1}^{+1} Q_n^\sigma \sin p\theta d\mu \quad \text{and} \quad \int_{-1}^{+1} Q_n^\sigma \cos p\theta d\mu,$$

we may either express Q_n^σ or the trigonometrical functions in terms of a series of powers of $\sin\theta$. The second alternative leads to results which in general are more convenient.

If we put

$$C_0 = 1; \qquad C_1 = p; \qquad C_2 = p\,.\frac{p}{2}; \qquad C_3 = p\,.\frac{p-1\,.\,p+1}{1\,.\,2\,.\,3};$$

$$C_\lambda = \frac{p}{\lambda\,!}\,\frac{(p+\lambda-2)\,!\,!}{(p-\lambda)\,!\,!};$$

$$B_0 = 1; \qquad B_1 = p; \qquad B_2 = \frac{p-1\,.\,p+1}{1\,.\,2}; \qquad B_3 = \frac{p-2\,.\,p\,.\,p+2}{1\,.\,2\,.\,3}$$

$$B_\lambda = \frac{1}{\lambda\,!}\,\frac{(p+\lambda-1)\,!\,!}{(p-\lambda-1)\,!\,!},$$

we find if σ be even, p odd, and n even,

$$\int Q_n \sin p\theta d\mu = \pi\,\frac{(n+\sigma-1)\,!\,!\,(n-3)\,!\,!}{(n-\sigma)\,!\,!\,(n+2)\,!\,!}\Big\{C_1\,.\,\sigma-1\,.\,\sigma+1$$

$$-\,C_3\,\frac{\sigma-3\,.\,\sigma-1\,.\,\sigma+1\,.\,\sigma+3}{n-3\,.\,n+4} + C_5\,\frac{\sigma-5\,.\,\sigma-3\,.\,\sigma-1\,.\,\sigma+1\,.\,\sigma+3\,.\,\sigma+5}{n-3\,.\,n-5\,.\,n+4\,.\,n+6}$$

$$-\,\ldots\ldots\Big\},$$

and if σ be even, p even, and n even,

$$= \pi\,\frac{(n+\sigma)\,!\,!\,(n-4)\,!\,!}{(n-\sigma-1)\,!\,!\,(n+3)\,!\,!}\Big\{B_1\,.\,\sigma-1\,.\,\sigma+1 - B_3\,\frac{\sigma-3\,.\,\sigma-1\,.\,\sigma+1\,.\,\sigma-3}{n-4\,.\,n+5}$$

$$+\,B_5\,\frac{\sigma-5\,.\,\sigma-3\,.\,\sigma-1\,.\,\sigma+1\,.\,\sigma+3\,.\,\sigma+5}{n-4\,.\,n-6\,.\,n+5\,.\,n+7} - \ldots\ldots\Big\}$$

If $n+\sigma+p$ be even, the integral is zero.

Similar equations are obtained for

$$\int_{-1}^{+1} Q_n^\sigma \cos p\theta d\mu.$$

7. The final results are expressed as follows :—

If g_n^σ denote the coefficient of $R_n^\sigma \cos\sigma\phi$ in the series of spherical harmonics when $R^\sigma = t_n^\sigma Q_n^\sigma$, and t_n^σ is a numerical coefficient, ϕ being

the longitude, the results of the investigation may be put into the form

$$g_n^\sigma = \sum_{p=0}^{p=n+1} a_p^\sigma r_n^\sigma, \text{ when } n \text{ is even, } \sigma \text{ odd, and } p \text{ odd.}$$

$$= \sum_{p=1}^{p=n+1} a_n^\sigma n_n^\sigma \quad ,, \quad n \;,,\; \text{odd, } \sigma \;,, \quad ,, \; p \text{ even.}$$

$$= \sum_{p=1}^{p=n+1} b_p^\sigma m_n^\sigma \quad ,, \quad n \;,,\; \text{even, } \sigma \text{ even, } ,, \; p \text{ odd.}$$

$$= \sum_{p=2}^{p=n+1} b_p^\sigma s_n^\sigma \quad ,, \quad n \;,,\; \text{odd, } \sigma \;,, \quad ,, \; p \text{ even.}$$

In these equations the factors $a_p^\sigma b_p^\sigma$ are the coefficients of the Fourier series (see § 2), and the quantities r_n^σ, n_n^σ, m_n^σ, s_n^σ are numerical quantities, which (as well as their logarithms) are given in tables at the end of the paper as far as $n = 12$, $\sigma = 12$, $p = 12$. By means of these tables the numerical work is reduced to a minimum, and the coefficients of the series may be obtained as far as terms of the 12th degree.

8. The proposed method is specially adapted to deal with problems like that of terrestrial magnetism, in which the function to be obtained as a series of spherical harmonics is not given directly, but by means of its differential coefficients. The force directed to the geographical north may by Fourier's analysis be obtained as a sum, the terms of which have the form $\cos \sigma\phi \cos p\theta$, and $\sin \sigma\phi \cos p\theta$ when σ is even, and the form $\cos \sigma\phi \sin p\theta$, $\sin \sigma\phi \sin p\theta$ when σ is odd. Integrating with respect to θ, the magnetic potential is obtained in a form such that the transformation into the series of spherical harmonics may be proceeded with. A separate expression of the magnetic potential is derived from the force directed to the geographical east.